CHARACTERIZATIONS OF CATEGORIES OF COMMUTATIVE C*-SUBALGEBRAS

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Abstract. We characterize the category of injective *-homomorphisms between commutative C*-subalgebras of various C*-algebras, namely C*-algebras of operators on separable Hilbert spaces, any finite-dimensional C*-algebra, and any commutative C*-algebra.

1. Introduction

The collection $\mathcal{C}(A)$ of commutative C*-subalgebras of a fixed C*-algebra A can be made into a category under various choices of morphisms. Two natural ones are inclusions and injective *-morphisms, resulting in categories $\mathcal{C}_{\subseteq}(A)$ and $\mathcal{C}_{\rightarrowtail}(A)$, respectively. Such categories are of paramount importance in the recent use of topos theory in research in foundations of physics, that proposes a new form of quantum logic [6, 12]. The goal of this article is to characterize which toposes are of the form studied in that programme. Eventually this should increase insight into the intrinsic structure of such toposes, and hence shed light on the foundations of quantum physics such toposes aim to model. Another motivation to study categories based on $\mathcal{C}(A)$ is the hope that they could lead to a noncommutative extension of Gelfand duality, or at least to interesting invariants of C*-algebras [2, 1].

Our main result is a characterization of $\mathcal{C}_{\rightarrowtail}(A)$ for type I factors A, as well as for finite-dimensional C*-algebras A and commutative C*-algebras A. This satisfactorily addresses a general theme in research in foundations of quantum mechanics. Specifically, it answers a categorification of *Piron's problem*, at least in the finite-dimensional case: which orthomodular lattices are those of closed subspaces of Hilbert space [18, 20, 16]? For choosing a commutative C*-subalgebra of the matrix algebra $M_n(\mathbb{C})$ amounts to choosing an orthonormal subset of $\mathbb{C}^n$; see also [11]. We also provide an appropriate generalization to countably infinite dimension.

Similarly, such a characterization has consequences in the study of *test spaces*. These are defined as collections of orthogonal subsets of a Hilbert space satisfying some conditions, and have been proposed as axioms for operational quantum mechanics. One of the major questions there is again which test spaces arise from propositions on Hilbert spaces [22].

The strategy behind our characterization is as follows. First, and this is the key insight, we recognize $\mathcal{C}_{\rightarrowtail}(D)$ for a commutative C*-algebra D as a certain amalgamation of a monoid M acting on a partially ordered set P. Such amalgamations

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are characterized in Section 3. Second, we use known results to characterize the partially ordered set $P = \mathcal{C}_{\subseteq}(D)$, consisting of partitions of the Gelfand spectrum of D, in Section 4. Third, we show that $\mathcal{C}_{\rightarrowtail}(A)$ is equivalent to $\mathcal{C}_{\rightarrowtail}(D)$ for a weakly terminal object D in $\mathcal{C}_{\rightarrowtail}(A)$ for the various types of C*-algebras A, finishing our characterization. This last step is the only one limiting our characterization to C*-algebras A that are type I factors, finite-dimensional, or commutative. Finally, Section 6 discusses the relation between $\mathcal{C}_{\rightarrowtail}$ and $\mathcal{C}_{\subseteq}$. Part of this discussion is already contained in the preliminaries of Section 2. Appendix A records some intermediate results of independent interest.

2. Preliminaries

Definition 1. Write $\mathcal{C}(A)$, or simply $\mathcal{C}$, for the collection of nonzero commutative C*-subalgebras C of a C*-algebra A. Here, we do not require C*-algebras to have a unit. This set of objects can be made into a category by various choices of morphisms, such as:

- inclusions $C \hookrightarrow C'$, given by $c \mapsto c$, yielding a (posetal) category $\mathcal{C}_{\subseteq}(A)$;
- injective *-morphisms $C \rightarrowtail C'$, giving a (left-cancellative) category $\mathcal{C}_{\rightarrowtail}(A)$.

The main theorem in the application of topos theory to foundations of quantum physics, that the canonical functor $C \mapsto C$ is an internal (possibly nonunital) C*-algebra [12, Theorem 6.4.8], holds in both toposes $\mathbf{Set}^{\mathcal{C}_{\subseteq}}$ and $\mathbf{Set}^{\mathcal{C}_{\rightarrowtail}}$ because of the fundamental Lemma 5 below. Categorically, $\mathcal{C}_{\rightarrowtail}$ is a more natural choice than $\mathcal{C}_{\subseteq}$, and Subsection 2.1 below argues that this choice is also more interesting from an algebraic point of view. Thus, our goal is to characterize toposes of the form $\mathbf{Set}^{\mathcal{C}_{\rightarrowtail}}$. The next theorem, due to Bunge, reduces this to characterizing the categories $\mathcal{C}_{\rightarrowtail}^{\mathrm{op}}$.

Theorem 2. *Let $\mathbf{T}$ be an elementary topos, and f the unique geometric morphism $\mathbf{T} \to \mathbf{Set}$, which has direct image part $f_* = \mathbf{T}(1, -)$. Then $\mathbf{T}$ is equivalent to $\mathrm{PSh}(\mathbf{C})$ if and only if there is a morphism $a \colon A \to I$ in $\mathbf{T}$ satisfying*

- *the canonical map $f^* f_* \forall_a (X \times A) \times_I A \to X$ is epic for each X in $\mathbf{T}$;*
- *the canonical map $E \times_{f_*(I)} f_* \forall_{a \times \mathrm{id}} (A \times A) \to f_* \forall_a ((f^*(E) \times_I A) \times A)$ is an isomorphism for each function $e \colon E \to f_*(I)$;*
- *if $g \colon X \to Y$ in $\mathbf{T}$ is epic, then so is $f_* \forall_a (g \times \mathrm{id})$;*

and $\mathbf{C} = f_*(A)$.

Proof. See [4]. □

2.1. **Invariants.** Let us temporarily consider von Neumann algebras A and their von Neumann subalgebras $\mathcal{V}(A)$, giving categories $\mathcal{V}_{\subseteq}$ and $\mathcal{V}_{\rightarrowtail}$. We will show that $\mathcal{V}_{\subseteq}$ contains exactly the same information as the projection lattice, so from that point of view $\mathcal{V}_{\rightarrowtail}$ is possibly more interesting. See also Remark 33 below. By extension, $\mathcal{C}_{\rightarrowtail}$ is possibly more interesting than $\mathcal{C}_{\subseteq}$ from this point of view, because $\mathcal{C}(A)$ and $\mathcal{V}(A)$ coincide for finite-dimensional C*-algebras A.

Denote the category of von Neumann algebras and unital normal *-homomorphisms by **Neumann**, and write **cNeumann** for the full subcategory of commutative algebras. Denote the category of orthomodular lattices and lattice morphisms preserving the orthocomplement by **OMLat**. The functor $\mathrm{Proj} \colon \mathbf{Neumann} \to \mathbf{OMLat}$ takes A to $\{p \in A \mid p^2 = p = p^*\}$ under the ordering $p \leq q$ iff $pq = p$. On morphisms $f \colon A \to B$ it acts as $p \mapsto f(p)$. Denote the essential image of Proj by $\mathbf{D}$; traditional quantum logic is the study of this subcategory of **OMLat** [19].

Denote by **Poset**[**cNeumann**] the category whose objects are sets of commutative von Neumann algebras C, partially ordered by inclusion (*i.e.* $C \leq C'$ iff $C \subseteq C'$), and whose morphisms are monotonic functions. We may regard $\mathcal{V}_{\subseteq}$ as a functor $\mathbf{Neumann} \to \mathbf{Poset}[\mathbf{cNeumann}]$. Denote the essential image of $\mathcal{V}_{\subseteq}$ by $\mathbf{C}$; this is a subcategory of **Poset**[**cNeumann**].

We now define two new functors, $F\colon \mathbf{C} \to \mathbf{D}$ and $G\colon \mathbf{D} \to \mathbf{C}$. The functor F acts on an object $\mathcal{V}_{\subseteq}(A)$ as follows. For each $C \in \mathcal{V}_{\subseteq}(A)$, we know that $\mathrm{Proj}(C)$ is a Boolean algebra [19, 4.16]. Because additionally the hypothesis of Kalmbach's Bundle lemma, recalled below, is satisfied, these Boolean algebras unite into an orthomodular lattice $F(\mathcal{V}_{\subseteq}(A))$. This assignment extends naturally to morphisms.

Lemma 3. *Let (B_i) be a family of Boolean algebras such that $\vee_i = \vee_j$, $\neg_i = \neg_j$, and $0_i = 0_j$ on intersections $B_i \cap B_j$. If $\leq$ on $\bigcup_i B_i$ is transitive, then $\bigcup_i B_i$ is an orthomodular lattice.*

Proof. See [15, 1.4.22]. □

The functor G acts on the projection lattice L of a von Neumann algebra as follows. Consider all complete Boolean sublattices B of L as a poset under inclusion. For each B, the continuous functions on its Stone spectrum form a commutative von Neumann algebra. Thus we obtain an object $G(L)$ in $\mathbf{C}$, and this assignment extends naturally to morphisms.

Theorem 4. *The objects* Proj *and* $\mathcal{V}_{\subseteq}$ *of* **Neumann/Cat** *are equivalent.*

$$\begin{tikzcd} & \mathbf{Neumann} \arrow[dl, "\mathcal{V}_{\subseteq}"'] \arrow[dr, "\mathrm{Proj}"] & \\ \mathbf{C} \arrow[rr, shift left, "F"] \arrow[rr, phantom, "\simeq"] & & \mathbf{D} \arrow[ll, shift left, "G"] \end{tikzcd}$$

Proof. Follows directly from the definitions and the previous lemma. □

Indeed, both $\mathcal{V}_{\subseteq}(A)$ and $\mathrm{Proj}(A)$ capture the Jordan algebra structure of A [10].

Returning to the setting of C*-algebras, notice that the previous theorem fails. There are C*-algebras without any projections, except for 0 and 1. But every C*-algebra has many commutative C*-subalgebras: every self-adjoint element generates one, and every element of a C*-algebra decomposes into a linear combination of self-adjoint elements. It might also be worth remarking that the functor $\mathcal{C}_{\subseteq}\colon \mathbf{Cstar} \to \mathbf{Poset}[\mathbf{cCstar}]$ factors through the category of partial C*-algebras [2].

2.2. **Functoriality.** The assignment $A \mapsto \mathcal{C}_{\subseteq}(A)$ extends to a functor: given a *-homomorphism $\varphi\colon A \to B$, direct images $C \mapsto \varphi(C)$ form a morphism of posets, for if $C \subseteq C'$, then $\varphi(C) \subseteq \varphi(C')$. Well-definedness relies on the following lemma.

Lemma 5. *The set-theoretic image of a C*-algebra under a *-homomorphism is again a C*-algebra.*

Proof. See [14, Theorem 4.1.9]. □

The assignment $A \mapsto \mathcal{C}_{\rightarrowtail}(A)$ has to be adapted to be made functorial. Either we only consider injective *-homomorphisms $A \rightarrowtail B$, or we restrict the target category $\mathcal{C}_{\rightarrowtail}(A)$ as follows.

Lemma 6. *There is a functor* $\mathbf{Cstar} \to \mathbf{Cat}$*, sending* A *to the subcategory of* $\mathcal{C}_{\rightarrowtail}(A)$ *with morphisms those* $i\colon C \to C'$ *satisfying*

$$i^{-1}(I \cap C') = I \cap C$$

for all closed (two-sided) ideals I *of* A*.*

Proof. Let $\varphi\colon A \to B$ be a *-homomorphism, and let i be as in the statement of the lemma. Then i induces a well-defined injective *-homomorphism $\varphi(C) \to \varphi(C')$ precisely when $\varphi(c_1) = \varphi(c_2) \iff \varphi(i(c_1)) = \varphi(i(c_2))$. Since φ and i are linear, this comes down to $\varphi(c) = 0 \iff \varphi(i(c)) = 0$, *i.e.* $\ker(\varphi) \cap C = \ker(\varphi \circ i)$. This becomes $I \cap C = i^{-1}(I \cap C')$ for $I = \ker(\varphi)$, and is therefore satisfied. □

Notice that when A is a topologically simple algebra, *e.g.* a matrix algebra, the subcategory of the previous lemma is actually the whole category $\mathcal{C}_{\rightarrowtail}(A)$.

3. Amalgamations

This section introduces the notion of a poset-monoid-amalgamation, and characterizes such categories. This is interesting in its own right, but even more so because it will turn out that $\mathcal{C}_{\rightarrowtail}$ is of this form. The main idea is to separate out symmetries into a monoid action, leaving just a partial order.

Definition 7. An *action* of a monoid M on a category $\mathbf{C}$ is a functor $F\colon M \to \mathbf{Cat}(\mathbf{C}, \mathbf{C})$. Write mx for the action of Fm on an object x of $\mathbf{C}$, and mf for the action of Fm on a morphism f of $\mathbf{C}$. The action is called *interpolative* when $\mathbf{C}(x, m_2 m_1 z) \neq \emptyset$ implies $\mathbf{C}(x, m_1 y) \neq \emptyset$ and $\mathbf{C}(y, m_2 z) \neq \emptyset$ for some object y.

Any action of a commutative monoid M on a partially ordered set P is interpolative: if $p \leq m_2 m_1 r$, taking $q = m_2 r$ gives $p \leq m_1 q$ and $q \leq m_2 r$.

Definition 8. If a monoid M acts on a category $\mathbf{C}$, then we can make a new category $\mathbf{C} \rtimes M$ whose objects are those of $\mathbf{C}$, and whose morphisms $x \to y$ are pairs (m, f) such that $\mathrm{dom}(f) = x$ and $\mathrm{cod}(f) = my$. Composition and identities are inherited from M and $\mathbf{C}$.

If the category $\mathbf{C}$ in the previous definition is a partially ordered set P, then $P \rtimes M$ has as objects $p \in P$, and morphisms $p \to q$ are $m \in M$ such that $p \leq mq$, with unit and composition from M.

An illustrative example to keep in mind is the following. Let M be the group of unitary n-by-n matrices. Let P be the lattice of subspaces of $\mathbb{C}^n$, ordered by inclusion. Then M acts on P. Morphisms in $P \rtimes M$ between subspaces $V \subseteq \mathbb{C}^n$ and $W \subseteq \mathbb{C}^n$ are unitary matrices U such that $U^{-1}(v) \in W$ for all $v \in V$.

This section characterizes categories of the form $P \rtimes M$ for an interpolative action of a monoid M on a poset P with a least element. Recall that a *retraction* of a functor is a left-inverse. An object 0 is *weakly initial* when for any object x there exists a (not necessarily unique) morphism $0 \to x$.

Lemma 9. *If a category* $\mathbf{A}$ *has a weak initial object* 0 *and a faithful retraction* F *of the inclusion* $\mathbf{A}(0,0) \to \mathbf{A}$*, then its objects are preordered by*

$$x \leq y \iff \exists f \in \mathbf{A}(x,y).\, F(f) = 1.$$

Proof. Clearly $\leq$ is reflexive, because $F(\mathrm{id}_x) = 1$. It is also transitive, for if $x \leq y$ and $y \leq z$, then there are $f\colon x \to y$ and $g\colon y \to z$ with $F(f) = 1 = F(g)$, so that $g \circ f\colon x \to z$ satisfies $F(g \circ f) = F(g) \circ F(f) = 1 \circ 1 = 1$ and $x \leq z$. □

Definition 10. A category $\mathbf{A}$ is called a *poset-monoid-amalgamation* when there exist a partial order P and a monoid M such that:

(A1) there is a weak initial object 0, unique up to isomorphism;
(A2) there is a faithful retraction F of the inclusion $\mathbf{A}(0,0) \to \mathbf{A}$;
(A3) there is an isomorphism $\alpha\colon \mathbf{A}(0,0) \to M$ of monoids;
(A4) there is an equivalence $(\mathbf{A}, \leq) \underset{\beta'}{\overset{\beta}{\rightleftarrows}} P$ of preorders;
(A5) for each object x there is an isomorphism $f\colon x \to \beta'(\beta(x))$ with $F(f) = 1$;
(A6) for each object y and $m\colon 0 \to 0$, there is $f\colon x \to y$ such that $F(f) = m$, and $f' = fg$ with $F(g) = 1$ for any $f'\colon x' \to y$ with $F(f') = m$;
(A7) if $F(f) = m_2 m_1$ for a morphism f, then $f = f_2 f_1$ with $F(f_i) = m_i$.

Example 11. If P is a partial order with least element, and M is a monoid acting interpolatively on P, then $P \rtimes M$ satisfies (A1)–(A7).

Proof. The least element 0 of P is a weak initial object, satisfying (A1). Conditions (A2)–(A4) are satisfied by definition, and (A5) is vacuous. To verify (A6) for $q \in P$ and $m \in M$, notice that $mq \leq mq$, and if $p \leq mq$, then certainly $p \leq 1mq$. Finally, (A7) is satisfied precisely because the action is interpolative. □

We can rephrase (A6) as: for each object y and morphism $m\colon 0 \to 0$, there is a greatest element of the set $\{f\colon x \to y \mid F(f) = m\}$, preordered by $f \leq g$ iff $f = hg$ for some morphism h satisfying $F(h) = 1$.

Lemma 12. *If $\mathbf{A}$ satisfies (A1)–(A7), then it induces an interpolative action of M on P given by $pm = \beta(x)$ if $f\colon x \to \beta'(p)$ is a greatest element with $\alpha(F(f)) = m$.*

Proof. First, notice that for any $p \in P$ and $m \in M$ there exists a greatest $f\colon x \to \beta'(p)$ with $\alpha(F(f)) = m$ by (A6). If there is another greatest $f'\colon x' \to \beta'(p)$ with $\alpha(F(f')) = m$, then there are morphisms $g\colon x \to x'$ and $g'\colon x' \to x$ with $F(g) = 1 = F(g')$. Hence $F(gg') = 1 = F(g'g)$, and because F is faithful, g is an isomorphism with g' as inverse. So $x \cong x'$, and therefore $\beta(x) \cong \beta(x')$. But because P is a partial order, this means $\beta(x) = \beta(x')$. thus the action is well-defined on objects.

To see that it is well-defined on morphisms, suppose that $p \leq q$. Then there is a morphism $f\colon \beta'(p) \to \beta'(q)$ with $F(f) = 1$. For any $m\colon 0 \to 0$, we can find maximal $f_p\colon x_p \to \beta'(p)$ with $F(f_p) = m$, and maximal $f_q\colon x_q \to \beta'(q)$ with $F(f_q) = m$. Now $ff_p\colon x_p \to \beta'(q)$ has $F(ff_q) = m$. Because f_q is a maximal such morphism, ff_p factors through f_q. That is, there is $h\colon x_p \to x_q$ with $f_q = ff_p h$ and $F(h) = 1$. So $mp \leq mq$.

Next, we verify that this assignment is functorial $G \to \mathbf{Cat}(P, P)$. Clearly $\mathrm{id}_{\beta'(p)}$ is maximal among morphisms $f\colon x \to \beta'(p)$ with $F(f) = 1$. Therefore $1p = \beta(\beta'(p)) = p$.

For $m_2, m_1 \in M$ and $p \in P$, we have $m_1 p = \beta(x_1)$ where $f_1\colon x_1 \to \beta'(p)$ is maximal with $\alpha(F(f_1)) = m_1$. So $m_2(m_1 p) = \beta(x_2)$ where $f_2\colon x_2 \to \beta'(\beta(x_1))$ is maximal with $\alpha(F(f_2)) = m_2$. By (A5), there is an isomorphism $h\colon x_1 \to \beta'(\beta(x_1))$ with $F(h) = 1$. So $h^{-1}f_2\colon x_2 \to x_1$ is maximal with $\alpha(F(h^{-1}f_2)) = m_2$. This gives $f = f_1 h^{-1} f_2\colon x_2 \to \beta'(p)$ with $\alpha(F(f)) = m_1 m_2$. If $g\colon y \to \beta'(p)$ has $\alpha(F(g)) = m_1 m_2$, then by (A7) there are $g_2\colon y \to z$ and $g_1\colon z \to \beta'(p)$ with

$g = g_1 g_2$ and $\alpha(F(g_i)) = m_i$.

$$\begin{tikzcd}
x_2 \arrow[r, "f_2"] \arrow[rrrr, bend left, "f"] & \beta'(\beta(x_1)) \arrow[rr, "h^{-1}"] & & x_1 \arrow[r, "f_1"] & \beta'(p) \\
& & z \arrow[ul, dashed, "hk"] \arrow[ur, dashed, "k"'] \arrow[urr, "g_1"] & & \\
y \arrow[uu, dashed, "l"] \arrow[urr, "g_2"] \arrow[uurrrr, bend right, "g"] & & & &
\end{tikzcd}$$

By maximality of f_1, there is a k with $g_1 = f_1 k$ and $\alpha(F(k)) = 1$. And by maximality of f_2, there is an l with $hkg_2 = f_2 l$ and $\alpha(F(l)) = 1$. Hence

$$g = g_1 g_2 = f_1 k g_2 = f_1 h^{-1} h k g_2 = f_1 h^{-1} f_2 l = fl.$$

So f is maximal with $F(f) = m_1 m_2$. Thus $(m_2 m_1)p = \beta(x_2) = m_2(m_1 p)$.

Finally, to see that the action is interpolative, suppose that $p \leq m_2 m_1 r$. Then there is $f\colon p \to m_2 m_1 r$ with $F(f) = 1$. By definition, $m_2 m_1 r = \beta(u)$ where $k\colon u \to \beta'(r)$ is maximal with $\alpha(F(k)) = m_2 m_1$. By (A7), $k = k_2 k_1$ for some k_i with $\alpha(F(k_i)) = m_i$. Say $y = \mathrm{cod}(k_1) = \mathrm{dom}(k_2)$, and take $q = \beta(y)$. By definition, $m_2 r = \beta(z_2)$ where $h_2\colon z_2 \to \beta'(r)$ is maximal with $\alpha(F(h_2)) = m_2$. Hence there is $f_2\colon y \to z_2$ such that $k_2 = h_2 f_2$, and $F(\beta(f_2)) = 1$. That is, $q \leq m_2 r$. Similarly, by definition $m_1 q = \beta(y_1)$, where $g_1\colon y_1 \to \beta'(q)$ is maximal with $\alpha(F(g_1)) = m_1$. Hence there is $f_1\colon u \to y_1$ such that $k_1 = g_1 f_1$ with $F(\beta(f_1)) = 1$. So the morphism $\beta(f_1)f\colon p \to m_1 q$ satisfies $F(\beta(f_1)f) = 1$. That is, $p \leq m_1 q$. □

Theorem 13. *If* $\mathbf{A}$ *satisfies (A1)–(A7), then there is an equivalence* $\mathbf{A} \to P \rtimes M$ *given by* $x \mapsto \beta(x)$ *on objects and* $f \mapsto \alpha(F(f))$ *on morphisms.*

Proof. First, it follows from (A6) that the assignment of the statement is well-defined, *i.e.* that $\alpha(F(f))$ is indeed a morphism of $P \rtimes M$. Indeed, if $f\colon x \to y$, then we need to show that $\beta(x) \leq \alpha(F(f)) \cdot \beta(y)$. Unfolding the definition of the action, this means we need to find a maximal $k\colon x' \to \beta'(\beta(y))$ with $F(f) = F(k)$, such that $\beta(x) \leq \beta(x')$. Unfolding the definition of the preorder, this means we need to find a morphism $h'\colon \beta'(\beta(x)) \to \beta'(\beta(x'))$ with $F(h') = 1$. By (A5), it suffices to find $h\colon x \to x'$ with $F(h) = 1$ instead. But by (A6), there exists a maximal $k\colon x' \to \beta'(\beta(y))$ with $F(k) = F(f)$. By its maximality, there exists $h\colon x \to x'$ with $F(h) = 1$ and $f = kh$. In particular, $\beta(x) \leq \beta(x')$.

Functoriality follows directly from the previous lemma, so indeed we have a well-defined functor $\mathbf{A} \to P \rtimes M$. Moreover, our functor is essentially surjective because β is an equivalence, and it is faithful because F is faithful.

Finally, to prove fullness, let $m\colon \beta(x) \to \beta(y)$ be a morphism in $P \rtimes M$. This means that $\beta(x) \leq \beta(y)m$, which unfolds to: there are a morphism $f\colon x \to z$ and a split monomorphism $h\colon z \to \beta'(\beta(y))$ in $\mathbf{A}$ with $F(f) = 1$ and h maximal with $\alpha(F(h)) = m$. By (A5), this is equivalent to the existence of a morphism $f\colon x \to z$ with $F(f) = 1$ and a morphism $h\colon z \to y$ in $\mathbf{A}$ maximal with $\alpha(F(h)) = m$. Now take $k = hf\colon x \to y$ in $\mathbf{A}$. Then

$$\alpha(F(k)) = \alpha(F(hf)) = \alpha(F(h))\alpha(F(f)) = m \cdot \alpha(1) = m \cdot 1 = m.$$

Hence our functor is full, and we conclude that it is (half of) an equivalence. □

Remark 14. If M is a group, we can replace (A6) and (A7) by the neater condition

(A6') for each y and $m\colon 0 \to 0$ there is an isomorphism $f\colon x \to y$ with $F(f) = m$.

Moreover, we do not need the action to be interpolative in this case. Instead of as in Lemma 12, the action is then recovered by $mp = \beta(x)$ when there is an isomorphism $f\colon x \to \beta'(p)$ with $\alpha(F(f)) = m$, and Theorem 13 still holds. This gives a neater characterization of amalgamations of group actions on posets with a least element. However, the monoid of interest in the appropriate infinite-dimensional setting is not a group, see Section 5 below.

4. Partition lattices

This section recalls a characterization of the partition lattice of a compact Hausdorff space due to Firby [7, 8]. This also gives a characterization of $\mathcal{C}_{\subseteq}(A)$ for commutative C*-algebras A.

An equivalence relation $\sim$ on a compact Hausdorff space X is *closed* when the set $\{x \in X \mid \exists u \in U.\, x \sim u\}$ is closed for every closed $U \subseteq X$. Closed equivalence relations on X, also called *partitions*, form a partial order $P(X)$ under *refinement*:

$$\sim \leq \approx \qquad \iff \qquad \big(\forall x, y \in X.\, x \sim y \implies x \approx y\big).$$

Notice that quotients of a compact Hausdorff space by an equivalence relation are again compact Hausdorff if and only if the equivalence relation is closed.

An element b of a lattice is called *bounding* when (i) it is zero or an atom; or (ii) it covers an atom and dominates exactly three atoms; or (iii) for distinct atoms p, q there exists an atom $r \leq b$ such that there are exactly three atoms less than $r \vee p$ and exactly three atoms less than $r \vee q$. A collection of atoms of a lattice with at least four elements is called *single* when it is a maximal collection of atoms of which the join of any two dominates exactly three atoms (not necessarily in the collection). A collection B of nonzero bounding elements of a lattice is called a *1-point* when (i) its atoms form a single collection; and (ii) if a is bounding and $a \geq b \in B$, then $a \in B$; and (iii) any $a \in B$ dominates an atom $p \in B$.

Theorem 15. *A lattice L with at least four elements is isomorphic to $P(X)$ for a compact Hausdorff space X if and only if:*

(P1) *L is complete and atomic;*

(P2) *the intersection of any two 1-points contains exactly one atom, and any atom belongs to exactly two 1-points;*

(P3) *for bounding $a, b \in L$ that are contained in a 1-point,*

$$\begin{aligned}&\{p \in \mathrm{Atoms}(L) \mid p \leq a \vee b\}\\ &= \{p \in \mathrm{Atoms}(L) \mid \textit{ if } x \textit{ is a 1-point with } p \in x \textit{ then } a \in x \textit{ or } b \in x\};\end{aligned}$$

for bounding $a, b \in L$ that are not contained in a 1-point,

$$\{p \in \mathrm{Atoms}(L) \mid p \leq a \vee b\} = \{p \in \mathrm{Atoms}(L) \mid p \leq a \textit{ or } p \leq b\};$$

(P4) *for 1-points $x \neq y$ there are bounding a, b with $a \notin x$, $b \notin y$, and $a \vee b = 1$;*

(P5) *joins of nests of bounding elements are bounding;*

(P6) *for nonzero $a \in L$, the collection B of bounding elements equal to or covered by a is the unique one satisfying:*
- *$\bigvee B = a$;*
- *no 1-point contains two members of B;*
- *if c is bounding, $b_1 \in B$, and no 1-point contains b_1 and c, then there is a bounding $b \geq c$ such that (i) there is no 1-point containing both*

b and b_1, and (ii) whenever there is a 1-point containing both b and $b_2 \in B$, then $b \geq b_2$;

(P7) *any collection of nonzero bounding elements that is not contained in a 1-point has a finite subcollection that is not contained in a 1-point;*

and X is (homeomorphic to) the set of 1-points of L, where a subset is closed if it is a singleton 1-point or it is the set of 1-points containing a fixed bounding element.

Proof. See [8]. □

Remark 16. The axiom responsible for compactness of X is (P7). The previous theorem holds for locally compact Hausdorff spaces X when we replace (P7) by

(P7') every 1-point contains a bounding b such that $\{l \in L \mid l \geq b\}$ satisfies (P7).

Indeed, because (P1)–(P6) already guarantee Hausdorffness, we may take local compactness to mean that every point has a compact neighbourhood that is closed. And closed sets correspond to sets of 1-points containing a fixed bounding element.

Corollary 17. *A lattice L is isomorphic to $\mathcal{C}_{\subseteq}(A)^{\mathrm{op}}$ for a commutative C*-algebra A of dimension at least three if and only if it satisfies (P1)–(P6) and (P7'). The C*-algebra A is unital if and only if L additionally satisfies (P7).*

Proof. The lattice $\mathcal{C}_{\subseteq}(A)$ is that of subobjects of A in the category of commutative (unital) C*-algebras and (unital) nondegenarate *-homomorphisms. Recall that a *subobject* is an equivalence class of monomorphisms into a given object, where two monics are identified when they factor through one another by an isomorphism. The dual notion is a *quotient*: an equivalence class of epimorphisms out of a given object. By Gelfand duality, $\mathcal{C}_{\subseteq}(A)$ is isomorphic to the opposite of the lattice of quotients of $X = \mathrm{Spec}(A)$. But the latter is precisely $P(X)^{\mathrm{op}}$, because categorical quotients in the category of (locally) compact Hausdorff spaces are quotient spaces. □

If X is compact and discrete, then $P(X)$ becomes the lattice of partitions of a finite set, and neater characterizations are available. In particular, the following alleviates the restriction in the previous theorem that L have at least four elements.

Recall that a lattice is *semimodular* if $a \vee b$ covers b whenever a covers $a \wedge b$. A lattice is *geometric* when it is atomic and semimodular. An element x in a lattice is *modular* when $a \vee (x \wedge y) = (a \vee x) \wedge y$ for all $a \leq y$. The *Möbius function* of a finite lattice is the unique function $\mu \colon L \to \mathbb{Z}$ satisfying $\sum_{y \leq x} \mu(x) = \delta_{0,x}$. It can be defined recursively by $\mu(0) = 1$ and $\mu(x) = -\sum_{y \leq x} \mu(x)$ for $x > 0$; see [3]. The *characteristic polynomial* of a finite lattice L is $\sum_{x \in L} \mu(x) \cdot \lambda^{\dim(1) - \dim(x)}$.

Theorem 18. *A lattice L is isomorphic to $P(\{1, \ldots, n+1\})$ if and only if:*

(F1) *it is geometric;*
(F2) *if $\mathrm{rank}(x) = \mathrm{rank}(y)$, then $\uparrow x \cong \uparrow y$;*
(F3) *it has a modular coatom;*
(F4) *its characteristic polynomial is $(\lambda - 1) \cdots (\lambda - n)$.*

Proof. See [23]. □

5. Weakly terminal subalgebras

Denote by $S(X)$ the monoid of continuous functions $f \colon X \twoheadrightarrow X$ with dense image on a locally compact Hausdorff space X.

Proposition 19. *For any locally compact Hausdorff space X, the monoid $S(X)$ acts interpolatively on $P(X)$ by*

$$(f\sim) \;=\; (f\times f)^{-1}(\sim).$$

Proof. First of all, notice that $f\sim$ is reflexive, symmetric and transitive, so indeed is a well-defined equivalence relation on X, which is closed because f is continuous. Concretely, $x(f\sim)y$ if and only if $f(x)\sim f(y)$. Moreover, clearly $\mathrm{id}\sim\,=\,\sim$, and $g(f\sim)=(gf)\sim$, so the above is a genuine action.

To see that the action is interpolative, let $\sim,\approxeq$ be equivalence relations on X, let $f,g\colon X\twoheadrightarrow X$ be continuous maps with dense image, and suppose $\sim\,\leq(gf\approxeq)$. Define $\approx$ as the closed equivalence relation on X generated by $(f\times f)(\sim)=\{(f(x),f(y))\mid x\sim y\}$. If $x\sim y$, then by definition $f(x)\approx f(y)$, so $x(f\approx)y$. In other words, $\sim\,\leq(f\approx)$, so there is a morphism $\sim\,\to(f\approx)$ in $P(X)$. If $x\approx y$, then there are $x_1,\ldots,x_n$ with $x=f(x_0)$, $y=f(x_n)$, $f(x_{2i})=f(x_{2i+1})$, and $x_{2i+1}\sim x_{2i+2}$. Hence $gf(x)\approxeq gf(x_1)\approxeq gf(x_2)\approxeq\cdots\approxeq gf(x_n)\approxeq gf(y)$. That is, there is a morphism $\approx\,\to(g\approxeq)$. □

Lemma 20. *If $A=C(X)$ for a locally compact Hausdorff space X, there is an isomorphism $\mathcal{C}_{\rightarrowtail}(A)^{\mathrm{op}}\cong P(X)\rtimes S(X)$ of categories.*

Proof. By definition, objects C of $\mathcal{C}_{\rightarrowtail}(A)$ are subobjects of $C(X)$ in the category of commutative C*-algebras. By Gelfand duality, these correspond to quotients of X in the category of locally compact Hausdorff spaces. But these, in turn, correspond to closed equivalence relations on X, establishing a bijection between the objects of $\mathcal{C}_{\rightarrowtail}(A)$ and $P(X)$.

A morphism $C\to C'$ in $\mathcal{C}_{\rightarrowtail}(A)$ corresponds through Gelfand duality to an epimorphism $g\colon Y'\twoheadrightarrow Y$ between the corresponding spectra. Writing the quotients as $Y=X/\sim$ and $Y'=X/\approx$ for closed equivalence relations $\sim$ and $\approx$, we find that g corresponds to a continuous $f\colon X\to X$ with dense image respecting equivalence:

$$x\approx y\implies f(x)\sim f(y).$$

But this just means $\approx\,\leq(\sim f)$. In other words, this is precisely a morphism $f\colon\approx\,\to\,\sim$ in $P(X)\rtimes S(X)$. □

Lemma 21. *If $\mathcal{C}_{\rightarrowtail}(A)$ has a weak terminal object D, then there is an equivalence $\mathcal{C}_{\rightarrowtail}(A)\simeq\mathcal{C}_{\rightarrowtail}(D)$ of categories.*

Proof. Clearly the inclusion $\mathcal{C}_{\rightarrowtail}(D)\hookrightarrow\mathcal{C}_{\rightarrowtail}(A)$ is a full and faithful functor, so it suffices to prove that it is essentially surjective. Let $C\in\mathcal{C}_{\rightarrowtail}(A)$. Then there exists an injective *-homomorphism $f\colon C\to D$ because D is weakly terminal. Hence $C\cong f(C)\in\mathcal{C}_{\rightarrowtail}(D)$. □

Lemma 22. *If $A=B(H)$ for a separable Hilbert space H, then $\mathcal{C}_{\rightarrowtail}(A)$ has a weak terminal object. The latter is *-isomorphic to:*

- $\ell^\infty(\{1,\ldots,n\})$ *when* $\dim(H)=n$;
- $L^\infty(0,1)\oplus\ell^\infty(\mathbb{N})$ *when H is infinite-dimensional.*

Proof. Let $C\in\mathcal{C}_{\rightarrowtail}(A)$. By Zorn's lemma, C is a C*-subalgebra of a maximal element of $\mathcal{C}_{\subseteq}(A)$. A maximal element in $\mathcal{C}_{\subseteq}(A)$ for a von Neumann algebra A is itself a von Neumann algebra, because it must equal its weak closure. It is known

that maximal abelian von Neumann subalgebras of $A = B(H)$ for an infinite-dimensional separable Hilbert space H are unitarily equivalent to one of the following: $L^\infty(0,1)$, $\ell^\infty(\{0,\ldots,n\})$ for $n \in \mathbb{N}$, $\ell^\infty(\mathbb{N})$, $L^\infty(0,1) \oplus \ell^\infty(\{0,\ldots,n\})$ for $n \in \mathbb{N}$, or $L^\infty(0,1) \oplus \ell^\infty(\mathbb{N})$ (see [14, Theorem 9.4.1]). Each of these allows an injective *-homomorphism into the latter one, $D = L^\infty(0,1) \oplus \ell^\infty(\mathbb{N})$. Therefore, there exists a morphism $C \to D$ in $\mathcal{C}_\rightarrowtail(A)$ for each C in $\mathcal{C}_\rightarrowtail(A)$, so that D is weakly terminal in $\mathcal{C}_\rightarrowtail(A)$.

If $\dim(H) = n$, then of the possibilities above, the maximal elements of $\mathcal{C}_\subseteq(A)$ can only be $\ell^\infty(\{1,\ldots,m\})$ for $m \leq n$. Hence, by a similar argument as for the infinite-dimensional case, $D = \ell^\infty(\{1,\ldots,n\})$ is weakly terminal in $\mathcal{C}_\rightarrowtail(A)$. □

The results of [21] indicate that the previous lemma might be extended to show that $\mathcal{C}_\rightarrowtail(A)$ has a weak terminal object for *any* von Neumann algebra A. But Theorem 15 only characterizes (locally) compact Hausdorff spaces, not the Gelfand spectra of von Neumann algebras, which are more specific hyperstonean spaces. We now arrive at our main result: the next theorem characterizes $\mathcal{C}_\rightarrowtail$ for type I factors.

Theorem 23. *For a category* $\mathbf{A}$*, the following are equivalent:*

- *the category* $\mathbf{A}$ *is equivalent to* $\mathcal{C}_\rightarrowtail(M_n(\mathbb{C}))^{\mathrm{op}}$*;*
- *the category* $\mathbf{A}$ *is equivalent to* $P(n) \rtimes S(n)$*;*
- *the following hold:*
  - $\mathbf{A}$ *satisfies (A1)–(A5) and (A6');*
  - $(\mathbf{A}, \leq)$ *satisfies (F1)–(F4) for* $n-1$*;*
  - $\mathbf{A}(0,0)$ *is isomorphic to the symmetric group on* n *elements.*

For a separable infinite-dimensional Hilbert space H*, the following are equivalent:*

- *the category* $\mathbf{A}$ *is equivalent to* $\mathcal{C}_\rightarrowtail(B(H))^{\mathrm{op}}$*;*
- *the category* $\mathbf{A}$ *is equivalent to* $P(X) \rtimes S(X)$*, where* X *is the topological space* $\mathrm{Spec}(L^\infty(0,1)) \sqcup \mathrm{Spec}(\ell^\infty(\mathbb{N}))$*;*
- *the following hold:*
  - $\mathbf{A}$ *satisfies (A1)–(A7);*
  - $(\mathbf{A}, \leq)$ *satisfies (P1)–(P7), giving a topological space* X*;*
  - $\mathbf{A}(0,0)$ *is isomorphic to the monoid* $S(X)$*;*
  - X *is homeomorphic to* $\mathrm{Spec}(L^\infty(0,1)) \sqcup \mathrm{Spec}(\ell^\infty(\mathbb{N}))$*.*

Proof. Combine the previous three lemmas with Theorem 13, and Theorem 18 or Theorem 15. The last condition in the infinite-dimensional case is the same as $X \cong \mathrm{Spec}(L^\infty(0,1) \oplus \ell^\infty(\mathbb{N}))$, because Gelfand duality turns products of C*-algebras into coproducts of compact Hausdorff spaces. □

The Gelfand spectrum of $\ell^\infty(\mathbb{N})$ is the Stone-Čech compactification of the discrete topology of $\mathbb{N}$. In other words, $\mathrm{Spec}(\ell^\infty(\mathbb{N}))$ consists of the ultrafilters on $\mathbb{N}$. A topological space is homeomorphic to $\mathrm{Spec}(L^\infty(0,1))$ if and only if it is compact, Hausdorff, totally disconnected, and its clopen subsets are isomorphic to the Boolean algebra of (Borel) measurable subsets of the interval $(0,1)$ modulo (Lebesgue) negligible ones. Notice that both spaces are compact, justifying the use of (P7) instead of (P7') in the previous theorem.

5.1. **Finite-dimensional C*-algebras.** In the finite-dimensional case, we can actually do better than factors: the next theorem characterizes $\mathcal{C}_\rightarrowtail(A)$ for any finite-dimensional C*-algebra A.

Lemma 24. *If $\mathcal{C}_{\rightarrowtail}(A_i)$ has a weak terminal object D_i for each i in a set I, then the C*-direct sum $\bigoplus_{i\in I} D_i$ is a weak terminal object in $\mathcal{C}_{\rightarrowtail}(\bigoplus_{i\in I} A_i)$.*

Proof. Let $C \in \mathcal{C}(\bigoplus_{i\in I} A_i)$. Then C is contained in the commutative subalgebra $\bigoplus_{i\in I} \pi_i(C)$ of $\bigoplus_{i\in I} A_i$. Because each D_i is weakly terminal, there exist morphisms $f_i \colon \pi_i(C) \to D_i$. Therefore $\bigoplus_{i\in I} f_i$ is a morphism $\bigoplus_{i\in I} \pi_i(C) \to \bigoplus_{i\in I} D_i$, and thus the latter is weakly terminal in $\mathcal{C}_{\rightarrowtail}(\bigoplus_{i\in I} A_i)$. □

Theorem 25. *A category $\mathbf{A}$ is equivalent to $\mathcal{C}_{\rightarrowtail}(A)^{\mathrm{op}}$ for a finite-dimensional C*-algebra A if and only if there are $n_1, \dots, n_k \in \mathbb{N}$ such that:*

- *$\mathbf{A}$ satisfies (A1)–(A5) and (A6');*
- *$(\mathbf{A}, \leq)$ satisfies (F1)–(F4) for $(\sum_{i=1}^k n_i) - 1$;*
- *$\mathbf{A}(0,0)$ is isomorphic to the symmetric group on $\sum_{i=1}^k n_i$ elements;*
- *$\sum_{i=1}^k n_i^2 = \dim(A)$.*

Proof. Every finite-dimensional C*-algebra A is of the form $\bigoplus_{i=1}^k M_{n_i}(\mathbb{C})$ with $n = \sum_{i=1}^k n_i^2$ [5, Theorem III.1.1]. By Lemmas 21, 22, and 24, we have

$$\mathcal{C}_{\rightarrowtail}(A) \simeq \mathcal{C}_{\rightarrowtail}(\bigoplus_{i=1}^k \mathbb{C}^{n_i}) \cong \mathcal{C}_{\rightarrowtail}(\mathbb{C}^{(\sum_{i=1}^k n_i)}).$$

So by Lemma 20, $\mathcal{C}_{\rightarrowtail}(A)^{\mathrm{op}} \simeq P(X) \rtimes S(X)$ for the discrete space X with $\sum_{i=1}^k n_i$ points. Now the statement follows from Theorems 13 and 18. □

6. Inclusions versus injections

This section compares $\mathcal{C}_{\subseteq}$ to $\mathcal{C}_{\rightarrowtail}$. For any category $\mathbf{C}$, recall that the category $\int_{\mathbf{C}} P$ of elements of a presheaf $P \in \mathrm{PSh}(\mathbf{C})$ is defined as follows. Objects are pairs (C, x) of $C \in \mathbf{C}$ and $x \in P(C)$. A morphism $(C,x) \to (D,y)$ is a morphism $f \colon C \to D$ in $\mathbf{C}$ satisfying $x = P(f)(y)$.

Lemma 26. *For any $P \in \mathrm{PSh}(\mathbf{C})$, the toposes $\mathrm{PSh}(\mathbf{C})/P$ and $\mathrm{PSh}(\int_{\mathbf{C}} P)$ are equivalent.*

Proof. See [17, Exercise III.8(a)]; we write out a proof for the sake of explicitness. Define a functor $F \colon \mathrm{PSh}(\mathbf{C})/P \to \mathrm{PSh}(\int_{\mathbf{C}} P)$ by

$$F\big(Q \overset{\alpha}{\Rightarrow} P\big)(C,x) = \alpha_C^{-1}(x),$$
$$F\big(Q \overset{\alpha}{\Rightarrow} P\big)\big((C,x) \xrightarrow{f} (D,y)\big) = Q(f),$$
$$F(Q \overset{\beta}{\Rightarrow} Q')_{(C,x)} = \beta_C.$$

Define a functor $G \colon \mathrm{PSh}(\int_{\mathbf{C}} P) \to \mathrm{PSh}(\mathbf{C})/P$ by $G(R) = (Q \overset{\alpha}{\Rightarrow} P)$ where

$$Q(C) = \coprod_{x \in P(C)} R(C,x),$$
$$Q(C \xrightarrow{f} D) = R\big((C, P(f)(y)) \xrightarrow{f} (D,y)\big),$$
$$\alpha_C(\kappa_x(r)) = x,$$

where $\kappa_x \colon R(C,x) \to \coprod_{x \in P(C)} R(C,x)$ is the coproduct injection. The functor G acts on morphisms as

$$G(R \overset{\beta}{\Rightarrow} R')_C = \coprod_{x \in P(C)} \beta_{(C,x)}.$$

Then one finds that $GF(Q \overset{\alpha}{\Rightarrow} P) = (Q \overset{\alpha}{\Rightarrow} P)$, and $FG(R) = \hat{R}$, where

$$\hat{R}(C,x) = \{x\} \times R(C,x),$$
$$\hat{R}\big((C,x) \xrightarrow{f} (D,y)\big) = \mathrm{id} \times R\big((C,P(f)(y)) \xrightarrow{f} (D,y)\big).$$

Thus there is a natural isomorphism $R \cong \hat{R}$, and F and G form an equivalence. $\square$

Definition 27. Define a presheaf $\mathrm{Aut} \in \mathrm{PSh}(\mathcal{C}_\rightarrowtail)$ by

$$\mathrm{Aut}(C) = \{i \colon C \xrightarrow{\cong} C' \mid C' \in \mathcal{C}\},$$
$$\mathrm{Aut}\big(C \overset{k}{\rightarrowtail} D\big)\big(j \colon D \xrightarrow{\cong} D'\big) = j\big|_{k(C)} \circ k \colon C \xrightarrow{\cong} j(k(C)).$$

Notice that $\mathrm{Aut}(C)$ contains the automorphism group of C. Also, any automorphism of A induces an element of $\mathrm{Aut}(C)$.

The category $\int_{\mathcal{C}_\rightarrowtail} \mathrm{Aut}$ of elements of Aut unfolds explicitly to the following. Objects are pairs (C,i) of $C \in \mathcal{C}$ and a *-isomorphism $i \colon C \xrightarrow{\cong} C'$. A morphism $(C,i) \to (D,j)$ is an injective *-homomorphism $k \colon C \rightarrowtail D$ such that $i = j \circ k$.

Lemma 28. *The categories $\mathcal{C}_\subseteq$ and $\int_{\mathcal{C}_\rightarrowtail} \mathrm{Aut}$ are equivalent.*

Proof. Define a functor $F \colon \mathcal{C}_\subseteq \to \int_{\mathcal{C}_\rightarrowtail} \mathrm{Aut}$ by $F(C) = (C, \mathrm{id}_C)$ on objects and $F(C \subseteq D) = (C \hookrightarrow D)$ on morphisms. Define a functor $G \colon \int_{\mathcal{C}_\rightarrowtail} \mathrm{Aut} \to \mathcal{C}_\subseteq$ by $G(C,i) = i(C) = \mathrm{cod}(i)$ on objects and $G\big(k \colon (C,i) \to (D,j)\big) = (i(C) \subseteq j(D))$ on morphisms. Then $GF(C) = C$, and $FG(C,i) = (i(C), \mathrm{id}_{i(C)}) \cong (C,i)$, so that F and G implement an equivalence. $\square$

Theorem 29. *The toposes $\mathrm{PSh}(\mathcal{C}_\subseteq)$ and $\mathrm{PSh}(\mathcal{C}_\rightarrowtail)/\mathrm{Aut}$ are equivalent.*

Proof. Combining the previous two lemmas, the equivalence is implemented explicitly by the functor $F \colon \mathrm{PSh}(\mathcal{C}_\rightarrowtail)/\mathrm{Aut} \to \mathrm{PSh}(\mathcal{C}_\subseteq)$ defined by

$$F\big(P \overset{\alpha}{\Rightarrow} \mathrm{Aut}\big)(C) = \alpha_C^{-1}(\mathrm{id}_C)$$
$$F\big(P \overset{\alpha}{\Rightarrow} \mathrm{Aut}\big)(C \subseteq D) = P(C \hookrightarrow D)$$

and the functor $G \colon \mathrm{PSh}(\mathcal{C}_\subseteq) \to \mathrm{PSh}(\mathcal{C}_\rightarrowtail)/\mathrm{Aut}$ defined by $G(R) = \big(P \overset{\alpha}{\Rightarrow} \mathrm{Aut}\big)$,

$$P(C) = \coprod_{i \colon C \xrightarrow{\cong} C'} R(i(C)),$$
$$P\big(C \overset{k}{\rightarrowtail} D\big) = \coprod_{j \colon D \xrightarrow{\cong} D'} R\big(j(k(C)) \subseteq j(D)\big),$$
$$\alpha_C(\kappa_i(r)) = i.$$

This proves the theorem. $\square$

Hence the topos $\mathrm{PSh}(\mathcal{C}_\rightarrowtail)$ is an *étendue*: the unique natural transformation from Aut to the terminal presheaf is (objectwise) epic, and the slice topos $\mathrm{PSh}(\mathcal{C}_\rightarrowtail)/\mathrm{Aut}$ is (equivalent to) a localic topos.

Lemma 30. *If $F\colon \mathbf{C} \to \mathbf{D}$ is (half of) an equivalence, X is any object of $\mathbf{C}$ and $Y \cong F(X)$, then the slice categories $\mathbf{C}/X$ and $\mathbf{D}/Y$ are equivalent.*

Proof. Let $G\colon \mathbf{D} \to \mathbf{C}$ be the other half of the given equivalence, and pick an isomorphism $i\colon Y \to F(X)$. Define a functor $H\colon \mathbf{C}/X \to \mathbf{D}/Y$ by $H(a\colon A \to X) = (i \circ Fa\colon FA \to Y)$ and $H(f\colon a \to b) = Ff$. Define a functor $K\colon \mathbf{D}/Y \to \mathbf{C}/X$ by $K(a\colon A \to Y) = (\eta_X^{-1} \circ Gi \circ Ga\colon GA \to X)$ and $K(f\colon a \to b) = Gf$. By naturality of η^{-1} we then have $KH(a) \cong a$. And because $G\varepsilon = \eta^{-1}$ we also have $HK(a) \cong a$. □

Lemma 31. *If the categories $\mathbf{C}$ and $\mathbf{D}$ are equivalent, then the toposes $\mathrm{PSh}(\mathbf{C})$ and $\mathrm{PSh}(\mathbf{D})$ are equivalent.*

Proof. Given functors $F\colon \mathbf{C} \to \mathbf{D}$ and $G\colon \mathbf{D} \to \mathbf{C}$ that form an equivalence, one directy verifies that $(-) \circ G\colon \mathrm{PSh}(\mathbf{C}) \to \mathrm{PSh}(\mathbf{D})$ and $(-) \circ F\colon \mathrm{PSh}(\mathbf{D}) \to \mathrm{PSh}(\mathbf{C})$ also form an equivalence. □

Theorem 32. *If $\mathcal{C}_{\rightarrowtail}(A)$ and $\mathcal{C}_{\rightarrowtail}(B)$ are equivalent categories, then $\mathcal{C}_{\subseteq}(A)$ and $\mathcal{C}_{\subseteq}(B)$ are Morita-equivalent posets, i.e. the toposes $\mathrm{PSh}(\mathcal{C}_{\subseteq}(A))$ and $\mathrm{PSh}(\mathcal{C}_{\subseteq}(B))$ are equivalent.*

Proof. If $\mathcal{C}_{\rightarrowtail}(A) \simeq \mathcal{C}_{\rightarrowtail}(B)$, then $\mathrm{PSh}(\mathcal{C}_{\rightarrowtail}(A)) \simeq \mathrm{PSh}(\mathcal{C}_{\rightarrowtail}(B))$ by Lemma 31. Moreover, the object Aut_B is (isomorphic to) the image of the object Aut_A under this equivalence. Hence

$$\mathrm{PSh}(\mathcal{C}_{\subseteq}(A)) \simeq \mathrm{PSh}(\mathcal{C}_{\rightarrowtail}(A))/\mathrm{Aut}_A \simeq \mathrm{PSh}(\mathcal{C}_{\rightarrowtail}(B))/\mathrm{Aut}_B \simeq \mathrm{PSh}(\mathcal{C}_{\subseteq}(B))$$

by Theorem 29. □

Remark 33. Hence $\mathcal{C}_{\rightarrowtail}(A)$ is an invariant of the topos $\mathrm{PSh}(\mathcal{C}_{\subseteq}(A))$ as well as of the C*-algebra A. It is not a complete invariant for the latter, however, as shown by Lemma 22. For example, $\mathcal{C}_{\rightarrowtail}(M_n(\mathbb{C})) \simeq \mathcal{C}_{\rightarrowtail}(\mathbb{C}^n)$, but $\mathcal{C}_{\subseteq}(M_n(\mathbb{C})) \not\cong \mathcal{C}_{\subseteq}(\mathbb{C}^n)$, and certainly $M_n(\mathbb{C}) \not\cong \mathbb{C}^n$.

We have relied heavily on equivalences of categories, and indeed a logical formula holds in the topos $\mathrm{PSh}(\mathbf{C})$ if and only if it holds in $\mathrm{PSh}(\mathbf{D})$ for equivalent categories $\mathbf{C}$ and $\mathbf{D}$. Therefore one might argue that $\mathcal{C}_{\rightarrowtail}$ has too many morphisms, as compared to $\mathcal{C}_{\subseteq}$, for toposes based on it to have internal logics that are interesting from the point of view of foundations of quantum mechanics. Instead of equivalences, one could consider isomorphisms of categories. This also resembles Piron's original question more closely. After all, an equivalence of partial orders is automatically an isomorphism. The following theorem shows that $\mathcal{C}_{\rightarrowtail}(A)$ is a weaker invariant of A than $\mathcal{C}_{\subseteq}(A)$, in this sense.

Theorem 34. *If $\mathcal{C}_{\rightarrowtail}(A)$ and $\mathcal{C}_{\rightarrowtail}(B)$ are isomorphic categories, then $\mathcal{C}_{\subseteq}(A)$ and $\mathcal{C}_{\subseteq}(B)$ are isomorphic posets.*

Proof. Let $K\colon \mathcal{C}_{\rightarrowtail}(A) \to \mathcal{C}_{\rightarrowtail}(B)$ be an isomorphism. Suppose that $C, D \in \mathcal{C}_{\rightarrowtail}(A)$ satisfy $C \subseteq D$. Consider the subcategory $\mathcal{C}_{\rightarrowtail}(D)$ of $\mathcal{C}_{\rightarrowtail}(A)$. On the one hand, by Lemma 20 it is isomorphic to $P(X) \rtimes S(X)$ for $X = \mathrm{Spec}(D)$, and therefore has a faithful retraction F_A of the inclusion $\mathcal{C}_{\rightarrowtail}(D) \to \mathcal{C}_{\rightarrowtail}(D)(0,0)$ by Theorem 13. On the other hand, K maps it to $\mathcal{C}_{\rightarrowtail}(K(D))$, which is isomorphic to $P(Y) \rtimes S(Y)$ for $Y = \mathrm{Spec}(K(D))$, and therefore similarly has a retraction F_B. Moreover, we have $KF_A = F_BK$. Now, by Theorem 13, inclusions in $\mathcal{C}_{\rightarrowtail}$ are characterized among all

morphisms f by $F(f) = 1$. Hence $F_B(K(C \hookrightarrow D)) = KF_A(C \hookrightarrow D) = K(1) = 1$, and therefore $K(C) \subseteq K(D)$. □

It remains open whether existence of an isomorphism $\mathcal{C}_{\subseteq}(A) \cong \mathcal{C}_{\subseteq}(B)$ implies existence of an isomorphism $\mathcal{C}_{\rightarrowtail}(A) \cong \mathcal{C}_{\rightarrowtail}(B)$. This question can be reduced as follows, because every injective *-morphism factors uniquely as a *-isomorphism followed by an inclusion. Supposing an isomorphism $F \colon \mathcal{C}_{\subseteq}(A) \to \mathcal{C}_{\subseteq}(B)$, we have $\mathcal{C}_{\rightarrowtail}(A) \cong \mathcal{C}_{\rightarrowtail}(B)$ if and only if there is an isomorphism $G \colon \mathcal{C}_{\cong}(A) \to \mathcal{C}_{\cong}(B)$ that coincides with F on objects. Now, in case A is (isomorphic to) $M_n(\mathbb{C})$, (so is B, and) if $C, D \in \mathcal{C}(A)$ are isomorphic then so are $F(C)$ and $F(D)$: if $C \cong D$, then $\dim(C) = \dim(D)$, so $\dim(F(C)) = \dim(F(D))$ because F preserves maximal chains, and hence $F(C) \cong F(D)$. However, it is not clear whether this behaviour is functorial, *i.e.* extends to a functor G.

Appendix A. Inverse semigroups and étendues

The direct proof of Theorem 29 follows from [13, A.1.1.7], but it can also be arrived at through a detour via inverse semigroups, based on results due to Funk [9]. This appendix describes the latter intermediate results, which might be of independent interest. Fix a unital C*-algebra A.

Definition 35. Define a set T with functions $T \times T \xrightarrow{\cdot} T$ and $T \xrightarrow{*} T$ by:

$$T = \left\{ C \overset{i}{\rightarrowtail} A \mid C \in \mathcal{C},\ i \text{ is an injective *-homomorphism} \right\},$$

$$(C' \overset{i'}{\rightarrowtail} A) \cdot (C \overset{i}{\rightarrowtail} A) = (i^{-1}(C') \overset{i' \circ i}{\rightarrowtail} A),$$

$$(C \overset{i}{\rightarrowtail} A)^* = (i(C) \overset{i^{-1}}{\rightarrowtail} A).$$

The multiplication is well-defined, because the inverse image of a *-algebra under a *-homomorphism is again a *-algebra, and the inverse image of a closed set is again a closed set, so that $i^{-1}(C)$ is indeed a commutative C*-algebra. The operation * is well-defined because of Lemma 5; and on the image, i^{-1} is a well-defined injective *-homomorphism. One can verify that together, these data form an inverse semigroup; that is, multiplication is associative, and i^* is the unique element with $ii^*i = i$ and $i^*ii^* = i^*$.

Lemma 36. *For $(C \overset{i}{\rightarrowtail} A) \in T$, we have $i^*i = (C \hookrightarrow A)$ and $ii^* = (i(C) \hookrightarrow A)$.*

Proof. For the former claim:

$$(C \overset{i}{\rightarrowtail} A)^* \cdot (C \overset{i}{\rightarrowtail} A) = (i(C) \overset{i^{-1}}{\rightarrowtail} A) \cdot (C \overset{i}{\rightarrowtail} A) = (i^{-1}(i(C)) \overset{i^{-1} \circ i}{\rightarrowtail} A) = (C \hookrightarrow A).$$

For the latter claim:

$$\begin{aligned}(C \overset{i}{\rightarrowtail} A) \cdot (C \overset{i}{\rightarrowtail} A)^* &= (C \overset{i}{\rightarrowtail} A) \cdot (i(C) \overset{i^{-1}}{\rightarrowtail} A) \\ &= ((i^{-1})^{-1}(C) \overset{i \circ i^{-1}}{\rightarrowtail} A) = (i(C) \hookrightarrow A).\end{aligned}$$

This proves the lemma. □

Definition 37. For any inverse semigroup T, one can define the groupoid $G(T)$ whose objects are the idempotents of T, i.e. the elements $e \in T$ with $e^2 = e$. A morphism $e \to f$ is an element $t \in T$ satisfying $e = t^*t$ and $tt^* = f$.

Proposition 38. *The groupoids $G(T)$ and $\mathcal{C}_{\cong}$ are isomorphic.*

Proof. An element $(C \overset{i}{\rightarrowtail} A)$ of T is idempotent when $i^{-1}(C) = C$ and $i^2 = i$ on C. That is, the objects of $G(T)$ are the inclusions $(C \hookrightarrow A)$ of commutative C*-subalgebras; we can identify them with $\mathcal{C}$.

A morphism $C \to C'$ in $G(T)$ is an element $(D \overset{j}{\rightarrowtail} A)$ of T such that $(C \hookrightarrow A) = j^*j = (D \hookrightarrow A)$ and $(C' \hookrightarrow A) = jj^* = (j(D) \hookrightarrow A)$, *i.e.* $C = D$ and $C' = j(D)$. That is, a morphism $C \to C'$ is an injective *-homomorphism $j \colon C \rightarrowtail C'$ that satisfies $j(D) = C'$, *i.e.* that is also surjective. In other words, a morphism $C \to C'$ is a *-isomorphism $C \to C'$. □

Definition 39. For any inverse semigroup T, one can define a partial order on the set $E(T) = \{e \in T \mid e^2 = e\}$ of idempotents by $e \leq f$ iff $e = fe$.

In fact, $G(T)$ is not an ordered groupoid, with $G(T)_0 = E(T)$.

Proposition 40. *The posets $E(T)$ and $\mathcal{C}_{\subseteq}$ are isomorphic.*

Proof. As with $G(T)$, objects of $E(T)$ can be identified with $\mathcal{C}$. Moreover, there is an arrow $C \to C'$ if and only if

$$(C \hookrightarrow A) = (C' \hookrightarrow A) \cdot (C \hookrightarrow A) = (C \cap C' \hookrightarrow A),$$

i.e. when $C \cap C' = C$. That is, there is an arrow $C \to C'$ iff $C \subseteq C'$. □

Also, $G(T)$ is always a subcategory of the following category $L(T)$.

Definition 41. For any inverse semigroup T, one can define the left-cancellative category $L(T)$ whose objects are the idempotents of T. A morphism $e \to f$ is an element $t \in T$ satisfying $e = t^*t$ and $t = ft$.

Proposition 42. *The categories $L(T)$ and $\mathcal{C}_{\rightarrowtail}$ are isomorphic.*

Proof. As with $G(T)$, objects of $L(T)$ can be identified with $\mathcal{C}$. A morphism $C \to C'$ in $L(T)$ is an element $(j \colon D \rightarrowtail A)$ of T such that $(C \hookrightarrow A) = j^*j = (D \hookrightarrow A)$ and

$$(D \overset{j}{\rightarrowtail} A) = (C' \hookrightarrow A) \cdot (D \overset{j}{\rightarrowtail} A) = (j^{-1}(C') \overset{j}{\rightarrowtail} A).$$

That is, a morphism $C \to C'$ is an injective *-homomorphism $j \colon C \rightarrowtail A$ such that $C = j^{-1}(C')$. Hence we can identify morphisms $C \to C'$ with injective *-homomorphisms $j \colon C \rightarrowtail C'$. □

Every ordered groupoid G has a classifying topos $\mathcal{B}(G)$. We now describe the topos $\mathcal{B}(G(T))$ explicitly, unfolding the definitions on [9, page 487].

For a presheaf $P \colon \mathcal{C}_{\subseteq}^{\mathrm{op}} \to \mathbf{Set}$, define another presheaf $P^* \colon \mathcal{C}_{\subseteq}^{\mathrm{op}} \to \mathbf{Set}$ by

$$P^*(C) = \{(j, x) \mid j \in \mathcal{C}_{\cong}(A)(C, C'), x \in P(C')\}.$$

On a morphism $C \subseteq D$, the presheaf $P^* \colon P^*(D) \to P^*(C)$ acts as

$$(k \colon D' \overset{\cong}{\to} D, y \in P(D')) \longmapsto \big(k\big|_C \colon C \overset{\cong}{\to} k(C),\, P(k(C) \subseteq D')(y)\big).$$

An object of $\mathcal{B}(G(T))$ is a pair (P, θ) of a presheaf $P \colon \mathcal{C}_{\subseteq}^{\mathrm{op}} \to \mathbf{Set}$ and a natural transformation $\theta \colon P^* \Rightarrow P$. A morphism $(P, \theta) \to (Q, \xi)$ is a natural transformation $\alpha \colon P \Rightarrow Q$ satisfying $\alpha \circ \theta = \xi \circ \alpha^*$, where the natural transformation $\alpha^* \colon P^* \Rightarrow Q^*$ is defined by $\alpha^*_C(j, x) = (j, \alpha_C(x))$.

Lemma 43. *The toposes $\mathrm{PSh}(\mathcal{C}_{\rightarrowtail})$ and $\mathcal{B}(G(T))$ are equivalent.*

Proof. Combine Proposition 42 with [9, Proposition 1.12]. Explicitly, (P, θ) in $\mathcal{B}(G(T))$ gets mapped to $F\colon \mathcal{C}_{\rightarrowtail}(A)^{\mathrm{op}} \to \mathbf{Set}$ defined by $F(C) = P(C)$ and

$$F(k\colon C \rightarrowtail D)(y) = \theta_C(k\colon C \overset{\cong}{\to} k(C),\, P(k(C) \subseteq D)(y)).$$

Conversely, F in $\mathrm{PSh}(\mathcal{C}_{\rightarrowtail})$ gets mapped to (P, θ), where

$$\begin{aligned} P(C) &= F(C), \\ P(C \subseteq D) &= F(C \hookrightarrow D), \\ \theta_C(j\colon C \overset{\cong}{\to} C', x \in F(C')) &= F(C' \overset{j^{-1}}{\to} C \subseteq D)(x). \end{aligned}$$

□

There is a canonical object $\mathbf{S} = (S, \theta)$ in $\mathcal{B}(G(T))$, defined as follows.

$$\begin{aligned} S(C) &= \{i\colon C \rightarrowtail A\}, \\ S(C \subseteq D)(j\colon D \rightarrowtail A) &= (j\big|_C\colon C \rightarrowtail A). \end{aligned}$$

In this case S^* becomes

$$\begin{aligned} S^*(C) &= \{(j, i) \mid j\colon C \overset{\cong}{\to} C', i\colon C' \rightarrowtail A\}, \\ S^*(C \subseteq D)(j, i) &= (j\mid_C\colon C \overset{\cong}{\to} j(C),\, i\big|_{j(C)}\colon j(C) \rightarrowtail A). \end{aligned}$$

Hence we can define a natural transformation $\theta\colon S^* \Rightarrow S$ by

$$\theta_C(j, i) = i \circ j.$$

The equivalence of the previous lemma maps $\mathbf{S}$ in $\mathcal{B}(G(T))$ to $\mathbf{D}$ in $\mathrm{PSh}(\mathcal{C}_{\rightarrowtail})$:

$$\begin{aligned} \mathbf{D}(C) &= \{i\colon C \rightarrowtail A\}, \\ \mathbf{D}(k\colon C \rightarrowtail D)(j\colon D \rightarrowtail A) &= (j \circ k\colon C \rightarrowtail A). \end{aligned}$$

Technically, the topos $\mathcal{B}(G(T))$ is an étendue: the unique morphism from some object $\mathbf{S}$ to the terminal object is epic, and the slice topos $\mathcal{B}(G(T))/\mathbf{S}$ is (equivalent to) a localic topos. The following lemma makes the latter equivalence explicit.

Lemma 44. *The toposes $\mathcal{B}(G(T))/\mathbf{S}$ and $\mathrm{PSh}(\mathcal{C}_{\subseteq})$ are equivalent.*

Proof. Combine Proposition 40 with equation (1) in [9, page 488]. □

Combining the previous two lemmas, we find:

Theorem 45. *The toposes $\mathrm{PSh}(\mathcal{C}_{\rightarrowtail})/\mathbf{D}$ and $\mathrm{PSh}(\mathcal{C}_{\subseteq})$ equivalent.* □

In our specific application, we have more information and it is helpful to reformulate things slightly. By Lemma 5, giving an injective *-homomorphism $i\colon C \rightarrowtail A$ is the same as giving a *-isomorphism $C \cong C'$ for some $C' \in \mathcal{C}$ (by taking $C' = i(C)$). Hence $\mathbf{S}$ is isomorphic to the object $\mathbf{Aut} = (\mathrm{Aut}, \theta)$ in $\mathcal{B}(G(T))$ with $\theta_C(j, i) = i \circ j$. This leads to Theorem 29.

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